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Solvability of complex Ginzburg-Landau equation in a general domain

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1 Introduction

In this paper we shall study the following complex Ginzburg-Landau equation in a general domain $\Omega\subset\mathbb{R}^{N}$ with smooth boundary $\partial\Omega$:

$$(\mathrm{CGL})\begin{cases}\partial_{t}u-(\lambda+i\alpha)\Delta u+(\kappa+i\beta)|u|^{q-2}u-\gamma u=f & \text{in } \Omega\times(0,\infty),\\ u=0 & \text{on } \partial\Omega\times(0,\infty),\\ u(x,0)=u_{0}(x), & x\in\Omega,\end{cases}$$

where $\lambda, \kappa\in\mathbb{R}_{+}:=(0,\infty)$, $\alpha,\beta,\gamma\in\mathbb{R}$ and $q\geq 2$ are constants; $i=\sqrt{-1}$ is the imaginary unit; $u_{0}:\Omega\rightarrow\mathbb{C}$ is an initial function; f : $\Omega\times(0,\infty)\rightarrow\mathbb{C}$ is an external force; $u:\overline{\Omega}\times[0,\infty)\rightarrow\mathbb{C}$ is a complex valued unknown function. In extreme cases, equation (CGL) includes two well-known equations: heat equation (when $\alpha=\beta=0$) and Schrödinger equation (when $\lambda=\kappa=0$). Thus we see that the equation (CGL) is "intermediate" between nonlinear heat and Schrödinger equations. From $\lambda>0$, we can regard (CGL) as a parabolic type equation, and from $\kappa>0$, we can fined that (CGL) has a negative feedback mechanism in the nonlinear term. By these insights, we can expect "smoothing effect" and "global solvability", respectively.

2 Notations and Preliminaries

In what follows, we identify $\mathbb{C}$ with $\mathbb{R}^{2}$: $u=u_{1}+iu_{2}\in\mathbb{C}\mapsto U=(u_{1}, u_{2})^{T}\in\mathbb{R}^{2}$.

$$\begin{aligned}&\mathbb{L}^{2}(\Omega):=\mathrm{L}^{2}(\Omega)\times \mathrm{L}^{2}(\Omega), && (U, V)_{\mathbb{L}^{2}}:=(u_{1}, v_{1})_{\mathrm{L}^{2}}+(u_{2}, v_{2})_{\mathrm{L}^{2}},\\ &\mathbb{L}^{q}(\Omega):=\mathrm{L}^{q}(\Omega)\times \mathrm{L}^{q}(\Omega), && |U|_{\mathbb{L}^{q}}^{q}:=|u_{1}|_{\mathrm{L}^{q}}^{q}+|u_{2}|_{\mathrm{L}^{q}}^{q},\\ &\mathbb{H}_{0}^{1}(\Omega):=\mathrm{H}_{0}^{1}(\Omega)\times \mathrm{H}_{0}^{1}(\Omega), && (U, V)_{\mathbb{H}_{0}^{1}}:=(u_{1}, v_{1})_{\mathrm{H}_{0}^{1}}+(u_{2}, v_{2})_{\mathrm{H}_{0}^{1}}.\end{aligned}$$

We introduce the following matrix I, which is a linear operator in $\mathbb{R}^{2}$ into itself:

$$I=\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}.$$

We use the nabla symbol $\nabla = (D_1, \ldots, D_N) : \mathbb{H}_0^1 \to (\mathrm{L}^2)^N \times (\mathrm{L}^2)^N$ as $\nabla U = (\nabla u_1, \nabla u_2)^T$. Then, the following properties are fundamental:

(i) Skew-symmetric property of the matrix I:

$$(IU \cdot V)_{\mathbb{R}^2} = -(U \cdot IV)_{\mathbb{R}^2}; \quad (IU \cdot U)_{\mathbb{R}^2} = 0 \quad \text{for each } U, V \in \mathbb{R}^2. \tag{2.1}$$

(ii) Commutative property of the matrix I and the differential opperator D_i:

$$ID_i = D_i I : \mathbb{H}_0^1 \to \mathbb{L}^2 \ (i = 1, \cdots, N). \tag{2.2}$$

(iii) Consequences from orthogonality of a vector V and IV:

$$(U \cdot V)_{\mathbb{R}^2}^2 + (U \cdot IV)_{\mathbb{R}^2}^2 = |U|_{\mathbb{R}^2}^2 |V|_{\mathbb{R}^2}^2 \quad \text{for each } U, V \in \mathbb{R}^2; \tag{2.3}$$

$$(U, V)_{\mathbb{L}^2}^2 + (U, IV)_{\mathbb{L}^2}^2 \leq |U|_{\mathbb{L}^2}^2 |V|_{\mathbb{L}^2}^2 \quad \text{for each } U, V \in \mathbb{L}^2(\Omega). \tag{2.4}$$

Now we define two functionals $\varphi, \psi : \mathbb{L}^2(\Omega) \to (-\infty, +\infty]$ by

$$\varphi(U) := \frac{1}{2} \int_\Omega |\nabla U(x)|_{\mathbb{R}^2}^2 dx \ \ (\text{if } U \in \mathbb{H}_0^1(\Omega)), \quad +\infty \ (\text{otherwise}), \tag{2.5}$$

$$\psi(U) := \frac{1}{q} \int_\Omega |U(x)|_{\mathbb{R}^2}^q dx \ \ (\text{if } U \in \mathbb{L}^q(\Omega) \cap \mathbb{L}^2(\Omega)), \quad +\infty \ (\text{otherwise}). \tag{2.6}$$

Then subdifferential of these functionals are, respectively, single valued and

$$\partial\varphi(U)(\cdot) = -\Delta U(\cdot) \quad (\text{where } \mathrm{D}(-\Delta) := \{U \in \mathbb{H}_0^1(\Omega) \mid \Delta U \in \mathbb{L}^2(\Omega)\}), \tag{2.7}$$

$$\partial\psi(U)(\cdot) = |U(\cdot)|_{\mathbb{R}^2}^{q-2} U(\cdot) \quad (\text{where } \mathrm{D}(|\cdot|_{\mathbb{R}^2}^{q-2} \cdot) := \mathbb{L}^{2(q-1)}(\Omega) \cap \mathbb{L}^2(\Omega)). \tag{2.8}$$

Proposition 2.1 (Brezis, H. [2] Theorem 9.). *Let B be maximal monotone and $\phi : \mathrm{H} \to \mathbb{R}_\infty$ be proper, convex and lower semi-continuous. Suppose*

$$\varphi((1 + \mu B)^{-1} u) \leq \varphi(u), \quad \forall \mu > 0, \ \forall u \in \mathrm{D}(\varphi). \tag{2.9}$$

Then $\partial\phi + B$ is maximal monotone.

Lemma 2.1. *If $\phi = \varphi$ and $B = \partial\psi$ given by* (2.5) *and* (2.8), *then the inequality* (2.9) *holds.*

Proof. Let $U \in \mathbb{C}_0^1(\Omega)$ and $V := (1+\mu\partial\psi)^{-1} U$. For a.e. $x \in \Omega$, $V(x) + \mu |V(x)|_{\mathbb{R}^2}^{q-2} V(x) = U(x)$. Thus defining $G : \mathbb{R}^2 \to \mathbb{R}^2 \ ; V \mapsto V + \mu |V|_{\mathbb{R}^2}^{q-2} V$, we have $G(V(x)) = U(x)$. Note that G is of class C^1 and bijective from $\mathbb{R}^2$ into itself, and its Jacobian determinant is given by

$$\det DG(V) = (1 + \mu |V|_{\mathbb{R}^2}^{q-2}) \{1 + \mu (q-1) |V|_{\mathbb{R}^2}^{q-2}\} \neq 0 \quad \text{for each } V \in \mathbb{R}^2.$$

Applying the inverse function theorem, we have $G^{-1} \in \mathrm{C}^1(\mathbb{R}^2; \mathbb{R}^2)$. Hence $V(x) = G^{-1}(U(x))$. This shows $(1 + \mu\partial\psi)^{-1} \mathbb{C}_0^1(\Omega) \subset \mathbb{C}_0^1(\Omega)$. Let $U \in \mathbb{H}_0^1(\Omega)$, $V := (1 + \mu\partial\psi)^{-1} U$ and $U_n \in \mathbb{C}_0^1(\Omega)$ satisfying $U_n \to U$ in $\mathbb{H}^1(\Omega)$. Let $V_n := (1 + \mu\partial\psi)^{-1} U_n \in \mathbb{C}_0^1(\Omega)$. Since

$$|V_n - V|_{\mathbb{L}^2} = |(1 + \mu\partial\psi)^{-1} U_n - (1 + \mu\partial\psi)^{-1} U|_{\mathbb{L}^2} \leq |U_n - U|_{\mathbb{L}^2} \to 0 \quad \text{as } n \to \infty,$$

we have $V_n \to V$ in $\mathbb{L}^2(\Omega)$. Also defferentiating $G(V_n(x)) = U_n(x)$ gives

$$(1 + \mu |V_n(x)|_{\mathbb{R}^2}^{q-2}) \nabla V_n(x) + \mu (q-2) |V_n(x)|_{\mathbb{R}^2}^{q-4} (V_n(x) \cdot \nabla V_n(x))_{\mathbb{R}^2} V_n(x) = \nabla U_n(x). \tag{2.10}$$

Multiplying (2.10) by $\nabla V_n(x)$, we have $|\nabla V_n(x)|^2_{\mathbb{R}^2} \le (\nabla U_n(x)\cdot\nabla V_n(x))_{\mathbb{R}^2}$. Therefore we have $|\nabla V_n|_{\mathbb{L}^2} \le |\nabla U_n|_{\mathbb{L}^2} \to |\nabla U|_{\mathbb{L}^2}$. Thus the boundedness of $\{\nabla V_n\}$ gives $V \in \mathbb{H}^1_0(\Omega)$, and we have $(1+\mu\partial\psi)^{-1}\mathrm{D}(\varphi) \subset \mathrm{D}(\varphi)$. In addition, by weak lower semi-continuity of the norm, we have $|\nabla V|_{\mathbb{L}^2} \le |\nabla U|_{\mathbb{L}^2}$. □

Now since the trivial inclusion $\lambda\partial\varphi + \kappa\partial\psi \subset \partial(\lambda\varphi+\kappa\psi)$ holds, we have shown

$$\lambda\partial\varphi + \kappa\partial\psi = \partial(\lambda\varphi+\kappa\psi) \quad \text{for all } \lambda,\kappa > 0. \tag{2.11}$$

Here, we can reduce (CGL) to the following evolution equation:

$$\text{(E)}\begin{cases} \frac{d}{dt}U(t)+\partial(\lambda\varphi+\kappa\psi)(U(t))+\alpha I\partial\varphi(U(t))+\beta I\partial\psi(U(t))-\gamma U(t) &= F(t), \quad t\in(0,\infty),\\ U(0) &= U_0. \end{cases}$$

We introduce the following region:

$$\mathrm{CGL}(r) := \left\{(x,y)\in\mathbb{R}^2 \;\middle|\; xy \ge 0 \text{ or } \frac{|xy|-1}{|x|+|y|} < r\right\}. \tag{2.12}$$

Also, we use the constant $c_q \in [0,\infty)$ which denotes a strength of the nonlinearity:

$$c_q := \frac{q-2}{2\sqrt{q-1}} \tag{2.13}$$

3 Main Results

Theorem 1. *Let $\Omega \subset \mathbb{R}^N$ be a general domain with smooth boundary, $F \in \mathrm{L}^2(0,T;\mathbb{L}^2(\Omega))$ for all $T>0$ and $(\frac{\alpha}{\lambda},\frac{\beta}{\kappa}) \in \mathrm{CGL}(c_q^{-1})$. If the initial value $U_0 \in \mathbb{H}^1_0(\Omega)\cap\mathbb{L}^q(\Omega)$, then there exists a solution $U \in \mathrm{C}([0,\infty);\mathbb{L}^2(\Omega))$ of the equation* (E) *satisfying*
(i) $U \in \mathrm{W}^{1,2}(0,T;\mathbb{L}^2(\Omega))$ *for all* $T>0$;
(ii) $U(t) \in \mathrm{D}(\partial\varphi)\cap\mathrm{D}(\partial\psi)$ *for a.e.* $t\in(0,\infty)$ *and satisfies* (E) *for a.e.* $t\in(0,\infty)$;
(iii) $\partial\varphi(U(\cdot))$, $\partial\psi(U(\cdot)) \in \mathrm{L}^2(0,T;\mathbb{L}^2(\Omega))$ *for all* $T>0$.

Theorem 2. *Let $\Omega \subset \mathbb{R}^N$ be a general domain with smooth boundary, $F \in \mathrm{L}^2(0,T;\mathbb{L}^2(\Omega))$ for all $T>0$ and $(\frac{\alpha}{\lambda},\frac{\beta}{\kappa}) \in \mathrm{CGL}(c_q^{-1})$. If the initial value $U_0 \in \mathbb{L}^2(\Omega)$, then there exists a solution $U \in \mathrm{C}([0,\infty);\mathbb{L}^2(\Omega))$ of the equation* (E) *satisfying*
(i) $U \in \mathrm{W}^{1,2}_{\mathrm{loc}}((0,\infty);\mathbb{L}^2(\Omega))$;
(ii) $U(t) \in \mathrm{D}(\partial\varphi)\cap\mathrm{D}(\partial\psi)$ *for a.e.* $t\in(0,\infty)$ *and satisfies* (E) *for a.e.* $t\in(0,\infty)$;
(iii) $\varphi(U(\cdot))$, $\psi(U(\cdot)) \in \mathrm{L}^1(0,T)$ *and* $t\varphi(U(t))$, $t\psi(U(t)) \in \mathrm{L}^\infty(0,T)$ *for all* $T>0$;
(iv) $\sqrt{t}\frac{d}{dt}U(t)$, $\sqrt{t}\partial\varphi(U(t))$, $\sqrt{t}\partial\psi(U(t)) \in \mathrm{L}^2(0,T;\mathbb{L}^2(\Omega))$ *for all* $T>0$.

4 Key Inequalities

Lemma 4.1. *The following inequalities hold for all $U \in \mathrm{D}(\partial\varphi)\cap\mathrm{D}(\partial\psi)$:*

$$|(\partial\varphi(U), I\partial\psi(U))_{\mathbb{L}^2}| \le c_q(\partial\varphi(U),\partial\psi(U))_{\mathbb{L}^2}, \tag{4.1}$$
$$|(\partial\varphi(U), I\partial\psi_\mu(U))_{\mathbb{L}^2}| \le c_q(\partial\varphi(U),\partial\psi_\mu(U))_{\mathbb{L}^2} \le c_q(\partial\varphi(U),\partial\psi(U))_{\mathbb{L}^2}, \tag{4.2}$$

where $\partial\psi_\mu(U) = \partial\psi((1+\mu\partial\psi)^{-1}U)$ is Yosida approximation of $\partial\psi(U)$.

Proof. Using the definition of Yosida approximation, and letting $V := (1+\mu\partial\psi)^{-1}U$, we can reduce (4.2) to (4.1). Thus it is enough to show (4.1).

Calculating the right-hand side of (4.1) by integration by parts, we have

$$(\partial\varphi(U), \partial\psi(U))_{\mathbb{L}^2} = \int_\Omega \left\{(q-2)|U|_{\mathbb{R}^2}^{q-4}|(U\cdot\nabla U)_{\mathbb{R}^2}|^2 + |U|_{\mathbb{R}^2}^{q-2}|\nabla U|_{\mathbb{R}^2}^2\right\}. \tag{4.3}$$

Also, by integration by parts with (2.1) and (2.2), the left-hand side of (4.1) becomes

$$\begin{aligned}(\partial\varphi(U), I\partial\psi(U))_{\mathbb{L}^2} &= (\nabla U, (q-2)|U|_{\mathbb{R}^2}^{q-4}(U\cdot\nabla U)_{\mathbb{R}^2}IU + |U|_{\mathbb{R}^2}^{q-2}I\nabla U)_{\mathbb{L}^2}\\ &= (q-2)\int_\Omega |U|_{\mathbb{R}^2}^{q-4}(U\cdot\nabla U)_{\mathbb{R}^2}\cdot(IU\cdot\nabla U)_{\mathbb{R}^2}.\end{aligned} \tag{4.4}$$

Thus by Young's inequality, (2.3) and (4.3), we obtain the desired (4.1) as follows.

$$\begin{aligned}|(\partial\varphi(U), I\partial\psi(U))_{\mathbb{L}^2}| &\le (q-2)\int_\Omega |U|_{\mathbb{R}^2}^{q-4}|(U\cdot\nabla U)_{\mathbb{R}^2}\cdot(IU\cdot\nabla U)_{\mathbb{R}^2}|\\ &\le (q-2)\int_\Omega |U|_{\mathbb{R}^2}^{q-4}\frac{1}{2\sqrt{q-1}}\left\{(q-1)|(U\cdot\nabla U)_{\mathbb{R}^2}|^2 + (IU\cdot\nabla U)_{\mathbb{R}^2}|^2\right\}\\ &= c_q\int_\Omega |U|_{\mathbb{R}^2}^{q-4}\left\{(q-2)|(U\cdot\nabla U)_{\mathbb{R}^2}|^2 + |U|_{\mathbb{R}^2}^2|\nabla U|_{\mathbb{R}^2}^2\right\}\\ &= c_q(\partial\varphi(U), \partial\psi(U))_{\mathbb{L}^2}.\end{aligned}$$

□

5 Solvability of Approximate Equation

We treat the following equation:

$$\text{(AE)}\begin{cases} \frac{d}{dt}U(t) + \partial(\lambda\varphi+\kappa\psi)(U(t)) + \alpha I\partial\varphi(U(t)) + B(U(t)) &= F(t), \quad t\in(0,\infty),\\ U(0) &= U_0,\end{cases}$$

where $B : \mathbb{L}^2(\Omega) \to \mathbb{L}^2(\Omega)$ is Lipschitz with Lipschitz constant L_B.

Proposition 5.1. *Let $\Omega \subset \mathbb{R}^N$ be a general domain, $F \in \mathrm{L}^2(0,T;\mathbb{L}^2(\Omega))$ for all $T > 0$, $\lambda, \kappa > 0$, $\alpha \in \mathbb{R}$ and $B : \mathbb{L}^2(\Omega) \to \mathbb{L}^2(\Omega)$ be Lipschitz. If $U_0 \in \mathbb{H}_0^1(\Omega) \cap \mathbb{L}^q(\Omega)$, then there exists a unique solution $U \in \mathrm{C}([0,\infty);\mathbb{L}^2(\Omega))$ of* (AE) *satisfying*
(i) *$U \in \mathrm{W}^{1,2}(0,T;\mathbb{L}^2(\Omega))$ for all $T > 0$;*
(ii) *$U(t) \in \mathrm{D}(\partial\varphi) \cap \mathrm{D}(\partial\psi)$ for a.e. $t \in (0,\infty)$ and satisfies* (AE) *for a.e. $t \in (0,\infty)$;*
(iii) *$\partial\varphi(U(\cdot))$, $\partial\psi(U(\cdot)) \in \mathrm{L}^2(0,T;\mathbb{L}^2(\Omega))$ for all $T > 0$.*

In order to prove Proposition 5.1, we approximate monotone perturbation term $\alpha I\partial\varphi(U)$ by $\alpha I\partial\varphi_\nu(U)$, where $\partial\varphi_\nu$ is Yosida approximation of $\partial\varphi$: $\partial\varphi_\nu(U) = \partial\varphi((1+\nu\partial\varphi)^{-1}U)$.

$$\text{(AE)}_\nu\begin{cases} \frac{d}{dt}U(t) + \partial(\lambda\varphi+\kappa\psi)(U(t)) + \alpha I\partial\varphi_\nu(U(t)) + B(U(t)) &= F(t), \quad t\in(0,\infty),\\ U(0) &= U_0.\end{cases}$$

Since $\alpha I\partial\varphi_\nu(\cdot)+B(\cdot)$ is Lipschitz in $\mathbb{L}^2(\Omega)$, approximate equation (AE)_ν has a unique solution $U = U_\nu \in \mathrm{C}([0,\infty);\mathbb{L}^2(\Omega))$ by the general theory of subdifferential operator (e.g. [2], [11]). Note that this approximate solution U_ν has the same regularities as those of the desired solution of Proposition 5.1. Then by the standard argument in the maximal monotone operator theory, we can show $\{U_\nu\}_{\nu\downarrow 0}$ is Cauchy in $C([0,T];\mathbb{L}^2(\Omega))$, as well as $\{\frac{d}{dt}U_{\nu_n}\}$, $\{\partial\varphi(U_{\nu_n})\}$ and $\{\partial\psi(U_{\nu_n})\}$ are bounded in $L^2(0,T;\mathbb{L}^2(\Omega))$. Hence by the demiclosedness of $\frac{d}{dt}$, $\partial\varphi$ and $\partial\psi$,

$$\begin{aligned}
&U_{\nu_n} \to U \quad \text{in } C([0,T];\mathbb{L}^2(\Omega));\\
&\frac{dU_{\nu_n'}}{dt} \rightharpoonup \frac{dU}{dt} \quad \text{in } L^2(0,T;\mathbb{L}^2(\Omega)),\\
&\partial\varphi(U_{\nu_n'}) \rightharpoonup \partial\varphi(U) \quad \text{in } L^2(0,T;\mathbb{L}^2(\Omega)),\\
&\partial\psi(U_{\nu_n'}) \rightharpoonup \partial\psi(U) \quad \text{in } L^2(0,T;\mathbb{L}^2(\Omega)),
\end{aligned}$$

for some sub sequence $\{\nu_n'\}_{n\in\mathbb{N}} \subset \{\nu_n\}_{n\in\mathbb{N}}$. Then by the definition of Yosida approximation,

$$\begin{aligned}
|U_{\nu_n} - J_{\nu_n}U_{\nu_n}|^2_{L^2(0,T;\mathbb{L}^2)} &= \int_0^T |U_{\nu_n}(s) - J_{\nu_n}U_{\nu_n}(s)|^2_{\mathbb{L}^2}ds\\
&= \nu_n^2\int_0^T |\partial\varphi_{\nu_n}(U_{\nu_n}(s))|^2_{\mathbb{L}^2}ds \le C_2\nu_n^2 \to 0 \quad \text{as } n\to\infty.
\end{aligned}$$

This means $J_{\nu_n}U_{\nu_n} \to U$ in $L^2(0,T;\mathbb{L}^2(\Omega))$. Now since $\partial\varphi_\nu(U_\nu) = \partial\varphi(J_\nu U_\nu)$, we have

$$\frac{dU}{dt} + \lambda\partial\varphi(U) + \kappa\partial\psi(U) + \alpha I\partial\varphi(U) + B(U) = F \quad \text{in } L^2(0,T;\mathbb{L}^2(\Omega)),$$

in the limit of the approximate equation $(\mathrm{AE})_{\nu_n'}$. That is, U is a desired solution of (AE).

6 Proof of Theorem 1

For the first step to prove Theorem 1, we approximate the equation (E) by

$$(\mathrm{E})_\mu \begin{cases} \frac{d}{dt}U(t)+\partial(\lambda\varphi+\kappa\psi)(U(t))+\alpha I\partial\varphi(U(t))+\beta I\partial\psi_\mu(U(t))-\gamma U(t) &= F(t), \quad t\in(0,\infty),\\ U(0) &= U_0, \end{cases}$$

where $\partial\psi_\mu(U) := \partial\psi((1+\mu\partial\psi)^{-1}U)$ is Yosida approximation of $\partial\varphi(U)$. This approximate equation $(\mathrm{E})_\mu$ is exactly the same form as that of (AE), whence by Proposition 5.1, $(\mathrm{E})_\mu$ has a solution $U = U_\mu \in C([0,\infty);\mathbb{L}^2(\Omega))$. Note that U_μ has the regularities stated in Proposition 5.1. In order to prove Theorem 1, we first derive some a priori estimates.

Lemma 6.1. *Let U be a solution of $(\mathrm{E})_\mu$. Fix $T>0$. Then there exists a positive constant C_1 depending only on γ, T, $|U_0|_{\mathbb{L}^2}$ and $\int_0^T|F|^2_{\mathbb{L}^2}$ satisfying*

$$\sup_{t\in[0,T]}|U(t)|^2_{\mathbb{L}^2} + \int_0^T\varphi(U(s))ds + \int_0^T\psi(U(s))ds \le C_1. \tag{6.1}$$

Proof. Multiplying $(\mathrm{E})_\mu$ by $U(t)$, we have, for a.e. $t\in(0,\infty)$,

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}|U(t)|^2_{\mathbb{L}^2} &+ 2\lambda\varphi(U(t)) + q\kappa\psi(U(t))\\
&+ \alpha(I\partial\varphi(U(t)),U(t))_{\mathbb{L}^2} + \beta(I\partial\psi_\mu(U(t)),U(t))_{\mathbb{L}^2}\\
&- \gamma|U(t)|^2_{\mathbb{L}^2} = (F(t),U(t))_{\mathbb{L}^2}.
\end{aligned}\tag{6.2}$$

Note that by integration by parts, (2.1) and (2.2), we have

$$\begin{aligned}
&(I\partial\varphi(U),U)_{\mathbb{L}^2} = 0,\\
&(I\partial\psi_\mu(U),U)_{\mathbb{L}^2} = (I\partial\psi(V), V+\mu\partial\psi(V))_{\mathbb{L}^2} = 0,
\end{aligned}$$

where $V := (1+\mu\partial\psi)^{-1}U$. Hence by (6.2) with Young's inequality, we have

$$\frac{1}{2}\frac{d}{dt}|U(t)|^2_{\mathbb{L}^2} + 2\lambda\varphi(U(t)) + q\kappa\psi(U(t)) \leq (\gamma_+ + \frac{1}{2})|U(t)|^2_{\mathbb{L}^2} + \frac{1}{2}|F(t)|^2_{\mathbb{L}^2}$$

where $\gamma_+ := \max\{\gamma, 0\}$. Thus the Gronwall's inequality yields

$$|U(t)|^2_{\mathbb{L}^2} + 2\int_0^t \{2\lambda\varphi(U(s)) + q\kappa\psi(U(s))\}\, ds \leq e^{(2\gamma_+ + 1)t}\left\{|U_0|^2_{\mathbb{L}^2} + \int_0^T |F|^2_{\mathbb{L}^2}\right\}$$

for all $t \in [0,T]$. Therefore we obtain the desired estiamte (6.1). □

Lemma 6.2. *Let U be a solution of $(\mathrm{E})_\mu$, and let $(\frac{\alpha}{\lambda}, \frac{\beta}{\kappa}) \in \mathrm{CGL}(c_q^{-1})$. Fix $T > 0$. Then there exist a positive constant C_2 depending only on $\lambda, \kappa, \alpha, \beta, \gamma$, $T, \varphi(U_0), \psi(U_0)$, $|U_0|_{\mathbb{L}^2}$ and $\int_0^T |F|^2_{\mathbb{L}^2}$ satisfying*

$$\sup_{t\in[0,T]} \varphi(U(t)) + \int_0^T \left|\frac{dU}{ds}\right|^2_{\mathbb{L}^2} ds + \int_0^T |\partial\varphi(U(s))|^2_{\mathbb{L}^2} ds + \int_0^T |\partial\psi(U(s))|^2_{\mathbb{L}^2} ds \leq C_2. \tag{6.3}$$

Proof. Let $V(t) := (1+\mu\partial\psi)^{-1}U(t)$. Since

$$(\partial\psi(U), \partial\psi_\mu(U))_{\mathbb{L}^2} = \int_\Omega |U|^{q-2}_{\mathbb{R}^2}|V|^{q-2}_{\mathbb{R}^2}(U\cdot V)_{\mathbb{R}^2} \geq \int_\Omega |V|^{2(q-1)}_{\mathbb{R}^2} = |\partial\psi_\mu(U)|^2_{\mathbb{L}^2};$$
$$(U, \partial\psi_\mu(U)) = q\psi(V) + \mu|\partial\psi(V)|^2_{\mathbb{L}^2} = q\psi_\mu(U) - (\frac{q}{2}-1)\mu|\partial\psi(V)|^2_{\mathbb{L}^2} \leq q\psi(U),$$

multiplying $(\mathrm{E})_\mu$ by $\partial\varphi(U(t))$ and $\partial\psi_\mu(U(t))$ yields

$$\frac{d}{dt}\varphi(U(t)) + \lambda|\partial\varphi(U(t))|^2_{\mathbb{L}^2} + \kappa G(t) + \beta B_\mu(t) = 2\gamma\varphi(U(t)) + (F, \partial\varphi(U(t)))_{\mathbb{L}^2}, \tag{6.4}$$
$$\frac{d}{dt}\psi_\mu(U(t)) + \kappa|\partial\psi_\mu(U(t))|^2_{\mathbb{L}^2} + \lambda G_\mu(t) - \alpha B_\mu(t) \leq q\gamma_+\psi(U(t)) + (F, \partial\psi_\mu(U(t)))_{\mathbb{L}^2}, \tag{6.5}$$

where $\gamma_+ := \max\{\gamma, 0\}$ and

$$\begin{cases} G := (\partial\varphi(U), \partial\psi(U))_{\mathbb{L}^2}, \\ G_\mu := (\partial\varphi(U), \partial\psi_\mu(U))_{\mathbb{L}^2}, \\ B_\mu := (\partial\varphi(U), I\partial\psi_\mu(U))_{\mathbb{L}^2}. \end{cases}$$

We add (6.4)$\times\delta^2$ and (6.5) for some $\delta > 0$ to get

$$\begin{aligned} \frac{d}{dt}&\{\delta^2\varphi(U) + \psi_\mu(U)\} + \delta^2\lambda|\partial\varphi(U)|^2_{\mathbb{L}^2} + \kappa|\partial\psi_\mu(U)|^2_{\mathbb{L}^2} \\ &+ \delta^2\kappa G + \lambda G_\mu + (\delta^2\beta - \alpha)B_\mu \\ &\qquad \leq \gamma_+\{2\delta^2\varphi(U) + q\psi(U)\} + (F, \delta^2\partial\varphi(U) + \partial\psi_\mu(U))_{\mathbb{L}^2}. \end{aligned} \tag{6.6}$$

Let $\epsilon \in (0, \min\{\lambda, \kappa\})$ be a small parameter. By the inequality of arithmetic and geometric means, and the fundamental property (2.4), we have

$$\begin{aligned} &\delta^2\lambda|\partial\varphi(U)|^2_{\mathbb{L}^2} + \kappa|\partial\psi_\mu(U)|^2_{\mathbb{L}^2} \\ &\quad = \epsilon\{\delta^2|\partial\varphi(U)|^2_{\mathbb{L}^2} + |\partial\psi_\mu(U)|^2_{\mathbb{L}^2}\} + (\lambda-\epsilon)\delta^2|\partial\varphi(U)|^2_{\mathbb{L}^2} + (\kappa-\epsilon)|\partial\psi_\mu(U)|^2_{\mathbb{L}^2} \\ &\quad \geq \epsilon\{\delta^2|\partial\varphi(U)|^2_{\mathbb{L}^2} + |\partial\psi_\mu(U)|^2_{\mathbb{L}^2}\} + 2\sqrt{(\lambda-\epsilon)(\kappa-\epsilon)\delta^2|\partial\varphi(U)|^2_{\mathbb{L}^2}|\partial\psi_\mu(U)|^2_{\mathbb{L}^2}} \\ &\quad \geq \epsilon\{\delta^2|\partial\varphi(U)|^2_{\mathbb{L}^2} + |\partial\psi_\mu(U)|^2_{\mathbb{L}^2}\} + 2\sqrt{(\lambda-\epsilon)(\kappa-\epsilon)\delta^2(G_\mu^2 + B_\mu^2)}. \end{aligned} \tag{6.7}$$

Note that by the key inequality Lemma 4.2

$$G \geq G_\mu \geq c_q^{-1}|B_\mu|. \tag{6.8}$$

Therefore combining (6.6), (6.7) and (6.8) yields

$$\begin{aligned}\frac{d}{dt}\left\{\delta^2\varphi(U)+\psi_\mu(U)\right\}+\epsilon\left\{\delta^2|\partial\varphi(U)|^2_{\mathbb{L}^2}+|\partial\psi_\mu(U)|^2_{\mathbb{L}^2}\right\}+J(\delta,\epsilon)|B_\mu|\\ \leq \gamma_+\left\{2\delta^2\varphi(U)+q\psi(U)\right\}+(F,\delta^2\partial\varphi(U)+\partial\psi_\mu(U))_{\mathbb{L}^2}.\end{aligned} \tag{6.9}$$

where

$$J(\delta,\epsilon):=2\delta\sqrt{(1+c_q^{-2})(\lambda-\epsilon)(\kappa-\epsilon)}+c_q^{-1}(\delta^2\kappa+\lambda)-|\delta^2\beta-\alpha|.$$

Now we show that $(\frac{\alpha}{\lambda},\frac{\beta}{\kappa})\in \mathrm{CGL}(c_q^{-1})$ gives $J(\delta,\epsilon)\geq 0$ for some δ and ϵ. By the continuity of $\epsilon\mapsto J(\delta,\epsilon)$ it suffices to show $J(\delta,0)>0$ for some δ. When $\alpha\beta>0$, it is enough to take $\delta=\sqrt{\alpha/\beta}$. When $\alpha\beta\leq 0$, we have $|\delta^2\beta-\alpha|=\delta^2|\beta|+|\alpha|$. Hence

$$J(\delta,0)=(c_q^{-1}\kappa-|\beta|)\delta^2+2\delta\sqrt{(1+c_q^{-2})\lambda\kappa}+(c_q^{-1}\lambda-|\alpha|).$$

Therefore if $|\beta|/\kappa\leq c_q^{-1}$, we have $J(\delta,0)>0$ for sufficiently large $\delta>0$. If $c_q^{-1}<|\beta|/\kappa$, we find that it is enough to see the descriminant is positive:

$$D/4:=(1+c_q^{-2})\lambda\kappa-(c_q^{-1}\kappa-|\beta|)(c_q^{-1}\lambda-|\alpha|)>0. \tag{6.10}$$

Since

$$D/4>0\Leftrightarrow \frac{|\alpha|}{\lambda}\frac{|\beta|}{\kappa}-1<c_q^{-1}\left(\frac{|\alpha|}{\lambda}+\frac{|\beta|}{\kappa}\right),$$

the condition $(\frac{\alpha}{\lambda},\frac{\beta}{\kappa})\in\mathrm{CGL}(c_q^{-1})$ yields $D>0$, whence $J(\delta,0)>0$ for some δ.

Now we take δ and ϵ satisfying $J(\delta,\epsilon)\geq 0$. By Lemma 6.1, integrating (6.9) gives

$$\sup_{t\in[0,T]}\varphi(U(t))+\int_0^T|\partial\varphi(U(s))|^2_{\mathbb{L}^2}ds+\int_0^T|\partial\psi_\mu(U(s))|^2_{\mathbb{L}^2}ds\leq C_2, \tag{6.11}$$

where C_2 depends on the constants stated in Lemma 6.2. We multiply $(\mathrm{E})_\mu$ by $\partial\psi(U)$ to get

$$\begin{aligned}&\frac{d}{dt}\psi(U)+\kappa|\partial\psi(U)|^2_{\mathbb{L}^2}+\lambda(\partial\varphi(U),\partial\psi(U))_{\mathbb{L}^2}\\ &\quad=-\alpha(I\partial\varphi(U),\partial\psi(U))_{\mathbb{L}^2}-\beta(I\partial\psi_\mu(U),\partial\psi(U))_{\mathbb{L}^2}+q\gamma\psi(U)+(F,\partial\psi(U))_{\mathbb{L}^2}\\ &\quad\leq\frac{\kappa}{4}|\partial\psi(U)|^2_{\mathbb{L}^2}+\frac{\alpha^2}{\kappa}|\partial\varphi(U)|^2_{\mathbb{L}^2}+q\gamma_+\psi(U)+\frac{\kappa}{4}|\partial\psi(U)|^2_{\mathbb{L}^2}+\frac{1}{\kappa}|F|^2_{\mathbb{L}^2}.\end{aligned} \tag{6.12}$$

Hence by (4.1) and (6.11), integrating (6.12) yields

$$\int_0^T|\partial\psi(U(s))|^2_{\mathbb{L}^2}ds\leq C_2. \tag{6.13}$$

Finally, combining $(\mathrm{E})_\mu$ with (6.11) and (6.13), we obtain the desired estimate (6.3). □

Now we prove Theorem 1.

Proof of Theorem 1. Let U_μ be a solution of $(\mathrm{E})_\mu$, and fix $T>0$. By Lemma 6.1 and 6.2, we have a sequence $\mu_n \downarrow 0$ satisfying

$$U_{\mu_n} \rightharpoonup U \quad \text{weakly in } \mathrm{L}^2(0,T;\mathbb{H}_0^1(\Omega)), \tag{6.14}$$
$$\frac{dU_{\mu_n}}{dt} \rightharpoonup \frac{dU}{dt} \quad \text{weakly in } \mathrm{L}^2(0,T;\mathbb{L}^2(\Omega)), \tag{6.15}$$
$$\partial\varphi(U_{\mu_n}) \rightharpoonup G \quad \text{weakly in } \mathrm{L}^2(0,T;\mathbb{L}^2(\Omega)), \tag{6.16}$$
$$\partial\psi(U_{\mu_n}) \rightharpoonup H \quad \text{weakly in } \mathrm{L}^2(0,T;\mathbb{L}^2(\Omega)), \tag{6.17}$$

for some function $G,H \in \mathrm{L}^2(0,T;\mathbb{L}^2(\Omega))$. Note that we use the weak closedness of $\frac{d}{dt}$ in $\mathrm{L}^2(0,T;\mathbb{L}^2(\Omega))$ to (6.15).

First we show $G=\partial\varphi(U)$ in $\mathrm{L}^2(0,T;\mathbb{L}^2(\Omega))$. For each $W \in \mathbb{C}_0^\infty(\Omega)$ and $w \in \mathrm{C}_0^\infty(0,T)$, we have $w(t)W \in \mathrm{L}^2(0,T;\mathbb{L}^2(\Omega))$. Hence in the limit of (6.14) and (6.16), we obtain

$$\int_0^T w(s)(G(s),W)_{\mathbb{L}^2}ds = \int_0^T w(s)(U(s),-\Delta W)_{\mathbb{L}^2}ds.$$

Then by the fandamental lemma of calculus of variations, $(G(t),W)_{\mathbb{L}^2} = (U(t),-\Delta W)_{\mathbb{L}^2}$ for a.e. $t\in(0,T)$, so that $-\Delta U(t) = G(t) \in \mathbb{L}^2(\Omega)$. Also by (6.14), $U(t) \in \mathbb{H}_0^1(\Omega)$ a.e. $t \in (0,T)$. Therefore $U(t) \in \mathrm{D}(\partial\varphi)$ and $\partial\varphi(U(t)) = -\Delta U(t) = G(t)$ for a.e. $t\in(0,T)$.

Next in order to see $H=\partial\psi(U)$ in $\mathrm{L}^2(0,T;\mathbb{L}^2(\Omega))$, we are showing

$$U_{\mu_n'} \to U \quad \text{in } \mathrm{C}(0,T;\mathbb{L}^2(\Omega')) \text{ for each bounded } \Omega' \subset \Omega, \tag{6.18}$$

for some subsequence $\{\mu_n'\} \subset \{\mu_n\}$. To confirm this, we use Ascoli's theorem and a diagonal argument. Let $\{\Omega_k\}_{k\in\mathbb{N}}$ be bounded domains in $\mathbb{R}^N$ with smooth boundaries satisfying (i) $\Omega_k \subset \Omega_{k+1} \subset \Omega$ for each $k \in \mathbb{N}$; (ii) for all bounded $\Omega' \subset \Omega$ there exists $k \in \mathbb{N}$ such that $\Omega' \subset \Omega_k$. Fix $k \in \mathbb{N}$. By Lemma 6.1 and 6.2, we have

$$|U_{\mu_n}(t_2) - U_{\mu_n}(t_1)|_{\mathbb{L}^2(\Omega_k)} \le \left\{\int_{t_1}^{t_2} \left|\frac{dU_{\mu_n}}{ds}\right|_{\mathbb{L}^2(\Omega)} ds\right\}^{\frac{1}{2}} \left\{\int_{t_1}^{t_2} ds\right\}^{\frac{1}{2}} \le \sqrt{C_2}\sqrt{t_2-t_1}, \tag{6.19}$$
$$|U_{\mu_n}(t)|^2_{\mathbb{H}^1(\Omega_k)} = |U_{\mu_n}(t)|^2_{\mathbb{L}^2(\Omega_k)} + |\nabla U_{\mu_n}(t)|^2_{\mathbb{L}^2(\Omega_k)} \le C_1 + 2C_2. \tag{6.20}$$

By (6.19), $\{U_{\mu_n}\}$ is uniformly equicontinuous in $\mathrm{C}(0,T;\mathbb{L}^2(\Omega_k))$, and by (6.20), $\{U_{\mu_n}(t)\}$ is relatively compact in $\mathbb{L}^2(\Omega)$ for each $t \in (0,T)$. Hence by Ascoli's theorem, we have

$$U_{\mu_n^k} \to U^k \quad \text{in } \mathrm{C}([0,T];\mathbb{L}^2(\Omega_k)) \text{ as } n\to\infty,$$

for some function $U^k \in \mathrm{C}([0,T];\mathbb{L}^2(\Omega_k))$ and some subsequence $\{\mu_n^k\}_{n\in\mathbb{N}} \subset \{\mu_n\}_{n\in\mathbb{N}}$. Now we take a subsequence successively from $k=1$ to ∞: $\{\mu_n^{k+1}\}_{n\in\mathbb{N}} \subset \{\mu_n^k\}_{n\in\mathbb{N}}$ for each $k\in\mathbb{N}$. Then the diagonal sequence $\{\mu_n^n\}_{n\in\mathbb{N}} =: \{\mu_n'\}_{n\in\mathbb{N}}$ satisfies

$$U_{\mu_n'} \to U^k \quad \text{in } \mathrm{C}([0,T];\mathbb{L}^2(\Omega_k)) \text{ as } n\to\infty \text{ for each } k\in\mathbb{N}. \tag{6.21}$$

On the other hand, by (6.14), we have

$$U_{\mu_n'} \rightharpoonup U \quad \text{weakly in } \mathrm{L}^2(0,T;\mathbb{L}^2(\Omega_k)) \text{ as } n\to\infty \text{ for each } k\in\mathbb{N}. \tag{6.22}$$

Thus by the uniqueness of a weak limit, we have $U^k = U$ in $\mathrm{L}^2(0,T;\mathbb{L}^2(\Omega_k))$. Finally since $\Omega' \subset \Omega_k$ for some k, we obtain the desired convergence (6.18) from (6.21).

Now we are show $H = \partial\psi(U)$ in $\mathrm{L}^2(0,T;\mathbb{L}^2(\Omega))$. By the demiclosedness of $U \mapsto |U|_{\mathbb{R}^2}^{q-2}U$ in $\mathrm{L}^2(0,T;\mathbb{L}^2(\Omega'))$, we have

$$U(t) \in \mathbb{L}^{2(q-1)}(\Omega') \quad \text{for a.e. } t \in (0,T), \tag{6.23}$$

$$H(t) = |U(t)|_{\mathbb{R}^2}^{q-2}U(t) \quad \text{in } \mathbb{L}^2(\Omega') \text{ for a.e. } t \in (0,T). \tag{6.24}$$

Since (6.24) holds for all bounded $\Omega' \subset \Omega$, we have $|U(t)|_{\mathbb{R}^2}^{q-2}U(t) = H(t)$ for a.e. $x \in \Omega$, so that $U(t) \in \mathrm{D}(\psi)$ and $H(t) = \partial\psi(U(t))$ for a.e. $t \in (0,T)$.

Finally we are showing that the function U satisfies equation (E). Note that $J_{\mu'_n}U_{\mu'_n} \to U$ in $\mathrm{L}^2(0,T;\mathbb{L}^2(\Omega'))$ by Lemma 6.2 where $J_\mu := (1+\mu\partial\psi)^{-1}$. By the demiclosedness of $\partial\psi$ in $\mathrm{L}^2(0,T;\mathbb{L}^2(\Omega'))$, we fined that U satisfies (E) in $\mathrm{L}^2(0,T;\mathbb{L}^2(\Omega'))$ for all bounded $\Omega' \subset \Omega$. Hence it also satisfies (E) in $\mathrm{L}^2(0,T;\mathbb{L}^2(\Omega))$. $U(0) = U_0$ in $\mathbb{L}^2(\Omega)$ can be obtained immediately from (6.18), since $U_{\mu'_n}(0) = U_0$ for each $n \in \mathbb{N}$. $\square$

7 Proof of Theorem 2

Now we are proving Theorem 2. Let $U_{0n} \in \mathbb{H}_0^1(\Omega) \cap \mathbb{L}^q(\Omega)$ satisfying $U_{0n} \to U_0$ in $\mathbb{L}^2(\Omega)$. By Theorem 1, we have a solution $U_n \in \mathrm{C}([0,T];\mathbb{L}^2(\Omega))$ corresponding to the initial value U_{0n}. First we derive some a priori estimates for the solution of (E) with $U_0 \in \mathbb{H}_0^1 \cap \mathbb{L}^q$.

Lemma 7.1. *Let U be a solution of* (E), *and fix $T > 0$. Then there exists a positive constant C_1 depending only on γ, T, $|U_0|_{\mathbb{L}^2}$ and $\int_0^T |F|_{\mathbb{L}^2}^2$ satisfying*

$$\sup_{t\in[0,T]} |U(t)|_{\mathbb{L}^2}^2 + \int_0^T \varphi(U(s))ds + \int_0^T \psi(U(s))ds \le C_1. \tag{7.1}$$

Lemma 7.2. *Let U be a solution of* (E) *with $U_0 \in \mathbb{H}_0^1(\Omega) \cap \mathbb{L}^q(\Omega)$ and $(\frac{\alpha}{\lambda}, \frac{\beta}{\kappa}) \in \mathrm{CGL}(c_q^{-1})$. Fix $T > 0$. Then there exist a positive constant C_2 depending only on $\lambda, \kappa, \alpha, \beta, \gamma, T$, $|U_0|_{\mathbb{L}^2}$ and $\int_0^T |F|_{\mathbb{L}^2}^2$ satisfying*

$$\sup_{t\in[0,T]} t\varphi(U(t)) + \int_0^T s\left|\frac{dU}{ds}\right|_{\mathbb{L}^2}^2 ds + \int_0^T s|\partial\varphi(U(s))|_{\mathbb{L}^2}^2 ds + \int_0^T s|\partial\psi(U(s))|_{\mathbb{L}^2}^2 ds \le C_2. \tag{7.2}$$

Since proofs are almost exactly the same as those of Lemma 6.1 and 6.2, we skip the details.

Proof of Theorem 2. Let U_n be a solution of (E) with $U_n(0) = U_{0n} \in \mathbb{H}_0^1(\Omega) \cap \mathbb{L}^q(\Omega)$, where $U_{0n} \to U_0$ in $\mathbb{L}^2(\Omega)$. By Lemma 7.1 and 7.2, we have $\{m_n\}_{n\in\mathbb{N}} \subset \{n\}_{n\in\mathbb{N}}$ satisfying

$$U_{m_n} \rightharpoonup U \quad \text{weakly in } \mathrm{L}^2_{\mathrm{loc}}((0,\infty);\mathbb{H}_0^1(\Omega)), \tag{7.3}$$

$$\sqrt{t}\frac{dU_{m_n}}{dt} \rightharpoonup \sqrt{t}\frac{dU}{dt} \quad \text{weakly in } \mathrm{L}^2(0,T;\mathbb{L}^2(\Omega)), \tag{7.4}$$

$$\sqrt{t}\partial\varphi(U_{m_n}) \rightharpoonup \sqrt{t}G \quad \text{weakly in } \mathrm{L}^2(0,T;\mathbb{L}^2(\Omega)), \tag{7.5}$$

$$\sqrt{t}\partial\psi(U_{m_n}) \rightharpoonup \sqrt{t}H \quad \text{weakly in } \mathrm{L}^2(0,T;\mathbb{L}^2(\Omega)), \tag{7.6}$$

for some function G, H. Note that we use the weak closedness of $\frac{d}{dt}$ in $L^2(\delta, T; \mathbb{L}^2(\Omega))$ for any $\delta \in (0, T)$ to (7.4). First by the same argument as those of Theorem 1, we have $G = \partial\varphi(U)$ in $L^2(\delta, T; \mathbb{L}^2(\Omega))$ for any $\delta \in (0, T)$, so that $G = \partial\varphi(U)$ a.e. $t \in (0, T)$. Next, also by the same argument as those of Theorem 1, we have

$$U_{m'_n} \to U \quad \text{in } C(\delta, T; \mathbb{L}^2(\Omega')) \text{ for each bounded } \Omega' \subset \Omega \text{ and } \delta \in (0, T), \tag{7.7}$$

for some subsequence $\{m'_n\} \subset \{m_n\}$. Therefore this yields $H = \partial\psi(U)$ in $L^2(\delta, T; \mathbb{L}^2(\Omega))$ for any $\delta \in (0, T)$, so that a.e. $t \in (0, T)$. Now we find that U satisfies equation (E) in the limit $(m'_n \to \infty)$ of the approximate equation of $U_{m'_n}$. Thus in order to finish the proof, it is enough to check

$$U(t) \to U_0 \quad \text{in } \mathbb{L}^2(\Omega) \text{ as } t \downarrow 0. \tag{7.8}$$

First we show $U(t) \rightharpoonup U_0$ weakly in $\mathbb{L}^2(\Omega)$. Multiplying the approximate equation of U_n by each $W \in \mathbb{C}_0^\infty(\Omega)$, we have

$$\begin{aligned} \frac{d}{dt}(U_n(t), W)_{\mathbb{L}^2} &= \gamma(U_n(t), W)_{\mathbb{L}^2} + (F(t), W)_{\mathbb{L}^2} \\ &\quad - ((\lambda + \alpha I)\partial\varphi(U_n(t)), W)_{\mathbb{L}^2} - ((\kappa + \beta I)\partial\psi(U_n(t)), W)_{\mathbb{L}^2}. \end{aligned} \tag{7.9}$$

Hence integrating (7.9) and taking the absolute value gives

$$\begin{aligned} |(U_n(t) - U_{0n}, W)_{\mathbb{L}^2}| &\leq |\gamma||W|_{\mathbb{L}^2} \int_0^t |U_n(s)|_{\mathbb{L}^2} ds + |W|_{\mathbb{L}^2} \int_0^t |F(s)|_{\mathbb{L}^2} ds \\ &\quad + (\lambda + |\alpha|)|\nabla W|_{\mathbb{L}^2} \int_0^t |\nabla U_n(s)|_{\mathbb{L}^2} ds \\ &\quad + (\kappa + |\beta|) \int_0^t \int_\Omega |U_n(s)|_{\mathbb{R}^2}^{q-1} |W|_{\mathbb{R}^2} dx ds. \end{aligned}$$

Thus using Hölder's inequality with Lemma 7.1, we have the estimate

$$\begin{aligned} |(U_n(t) - U_{0n}, W)_{\mathbb{L}^2}| &\leq |\gamma|\sqrt{C_1}|W|_{\mathbb{L}^2} t + \left\{ \int_0^t |F(s)|_{\mathbb{L}^2}^2 ds \right\}^{\frac{1}{2}} |W|_{\mathbb{L}^2} t^{\frac{1}{2}} \\ &\quad + (\lambda + |\alpha|)\sqrt{2C_1}|\nabla W|_{\mathbb{L}^2} t^{\frac{1}{2}} + (\kappa + |\beta|)(qC_1)^{\frac{q-1}{q}} |W|_{\mathbb{L}^q} t^{\frac{1}{q}}. \end{aligned} \tag{7.10}$$

Letting $n = m'_n \to \infty$, we have $|(U(t) - U_0, W)_{\mathbb{L}^2}| \leq Ct^{\frac{1}{q}}$ for sufficiently small $t > 0$, so that $U(t) \to U_0$ in $\mathcal{D}'(\Omega)$. Since $\mathbb{C}^\infty(\Omega) \subset \mathbb{L}^2(\Omega)$ is dense, we have $U(t) \rightharpoonup U_0$ weakly in $\mathbb{L}^2(\Omega)$.

Then we show $|U(t)|_{\mathbb{L}^2}^2 \to |U_0|_{\mathbb{L}^2}^2$. By the argument of Lemma 7.1, we have

$$|U_n(t)|_{\mathbb{L}^2}^2 \leq e^{(2\gamma_+ + 1)t} \left\{ |U_{0n}|_{\mathbb{L}^2}^2 + \int_0^t |F(s)|_{\mathbb{L}^2}^2 ds \right\}.$$

Hence letting $n \to \infty$ gives $|U(t)|_{\mathbb{L}^2}^2 \leq e^{(2\gamma_+ + 1)t} \{|U_0|_{\mathbb{L}^2}^2 + \int_0^t |F(s)|_{\mathbb{L}^2}^2 ds\}$. Then letting $t \downarrow 0$, we have $\overline{\lim}_{t\downarrow 0}|U(t)|_{\mathbb{L}^2}^2 \leq |U_0|_{\mathbb{L}^2}^2$. On the other hand, since $U(t) \rightharpoonup U_0$, we have $|U_0|_{\mathbb{L}^2}^2 \leq \underline{\lim}_{t\downarrow 0}|U(t)|_{\mathbb{L}^2}^2$ by the weak lower semicontinuity of the norm. Therefore $|U(t)|_{\mathbb{L}^2}^2 \to |U_0|_{\mathbb{L}^2}^2$. $\square$

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